MAXIMAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH LOCALLY INTEGRABLE FORCING TERM*

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ABSTRACT

We study the existence of a maximal solution of $-\Delta u + g(u) = f(x)$ in a domain $\Omega \in \mathbb{R}^N$ with compact boundary, assuming that $f \in (L^1_{loc}(\Omega))_+$ and that g is nondecreasing, $g(0) \geq 0$ and g satisfies the Keller–Osserman condition. We show that if the boundary satisfies the classical $C_{1,2}$ Wiener criterion, then the maximal solution is a large solution, i.e., it blows up everywhere on the boundary. In addition, we discuss the question of uniqueness of large solutions.

1. Introduction

Let Ω denote a subdomain of $\mathbb{R}^N$, $N \geq 2$, $\rho_{\partial\Omega}(x) = \text{dist}(x, \partial\Omega), \forall x \in \mathbb{R}^N$, and let $g \in C(\mathbb{R})$ be nondecreasing. In a preceding article [10], we studied existence and uniqueness of solutions of the problem

$$-\Delta u + g(u) = 0 \quad \text{in } \Omega, \tag{1.1}$$

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subject to the boundary blow-up condition

$$\lim_{\substack{\rho_{\partial\Omega}(x)\to 0\\ x\in K}} u(x)=\infty \quad \forall K\subset\Omega, \quad K \text{ bounded}. \tag{1.2}$$

Such a function u is called a **large solution**. In this article we extend the study to the equation with forcing term,

$$-\Delta u+g(u)=f(x) \quad \text{in } \Omega, \tag{1.3}$$

where $f\in L^1_{loc}(\Omega)$ is nonnegative. We assume throughout the paper that g satisfies the following conditions:

$$g\in C(\mathbb{R}), \quad g \text{ nondecreasing}, \quad g(0)\geq 0. \tag{1.4}$$

By a solution of (1.3) we mean a locally integrable function u such that $g(u)\in L^1_{loc}(\Omega)$ and (1.3) holds in the distribution sense. Accordingly, if u is a solution of (1.3), then $\Delta u\in L^1_{loc}(\Omega)$ and consequently $u\in W^{1,p}_{loc}(\Omega)$ for some $p>1$ (see [4]). Therefore, if Ω' is a smooth bounded domain such that $\bar{\Omega}'\subset\Omega$, then u possesses an L^1 trace on $\partial\Omega'$ and, if ϕ is a function in $C^2_0(\bar{\Omega}')$, i.e., $\phi\in C^2(\bar{\Omega}')$ and $\phi=0$ on $\partial\Omega'$, then

$$\int_{\Omega'}(-u\Delta\phi+g(u)\phi)dx=\int_{\Omega'}f\phi dx-\int_{\partial\Omega'}u\partial\phi/\partial\mathbf{n}'dS, \tag{1.5}$$

where $\mathbf{n}'$ denotes the external unit normal on $\partial\Omega'$. The boundary blow-up condition should be understood as an essential limit: u is bounded below a.e. by a function u_0 which satisfies (1.2).

In a well-known paper [3] Brezis proved that, for any $q>1$ and $f\in L^1_{loc}(\mathbb{R}^N)$, there exists a unique solution of the equation

$$-\Delta u+|u|^{q-1}u=f \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

The proof was based upon a duality argument which implied local $L^q_{loc}(\mathbb{R}^N)$-bounds of approximate solutions.

In the present paper we investigate this problem, for $f\geq 0$, for a large family of nonlinearities and arbitrary domains with compact boundary satisfying a mild regularity assumption. When $\Omega\subsetneqq\mathbb{R}^N$, we shall concentrate on the existence and uniqueness of *large solutions*, i.e., solutions which blow up on the boundary. Other boundary value problems may have no solution when $f\in(L^1_{loc})_+$. For instance, if Ω is a smooth, bounded domain and the boundary data is in $L^1(\partial\Omega)$, then the boundary value problem for (1.3) possesses a solution (in the L^1 sense) if and only if $f \in L^1(\Omega;\rho)$, where $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. In fact, in this case, if $f \in C(\Omega)$ and $f \geq c_0\rho^{-2}$ for some positive constant c_0, then every solution u of (1.3), such that $u \geq 0$ in a neighborhood of the boundary, is necessarily a large solution. However, one can establish a partial result, namely, the existence of a *minimal solution* of the equation which is also a supersolution of the boundary value problem, (see Theorem 1.2 below).

The problem of existence of large solutions is closely related to the question of existence of *maximal solutions*. A maximal solution (if it exists) need not be a large solution. It is well known that, for equation (1.6) with $f = 0$, a maximal solution exists in any domain. This is a consequence of the estimates of Keller [5] and Osserman [12] as it was shown in [7]. In a recent paper, Labutin [6] presented a necessary and sufficient condition on Ω for the maximal solution of (1.6) with $f = 0$ to be a large solution.

A function g satisfies the Keller–Osserman condition (see [5] and [12]) if for every $a > 0$

$$\int_a^\infty \left(\int_0^t g(s)ds\right)^{-1/2} dt < \infty. \tag{1.7}$$

Our first result concerns the existence of maximal solutions.

THEOREM 1.1: *Let Ω be a domain in $\mathbb{R}^N$ and let g be a function satisfying (1.4) and the Keller–Osserman condition. In addition, assume that (1.1) possesses a subsolution. Then (1.3) possesses a maximal solution, for every nonnegative $f \in L^1_{loc}(\Omega)$.*

Remark: If Ω is bounded or if $g(r_0) = 0$ for some $r_0 \in \mathbb{R}$, then equation (1.1) possesses a solution. In fact it possesses a *bounded* solution. If g remains positive and the domain is unbounded, some conditions for the existence of a solution of (1.1) can be found in [10].

The existence of a maximal solution implies that the family of all solutions of (1.3) is locally uniformly bounded from above. By [5] and [12] the Keller–Osserman condition is *necessary* for this property to hold. Furthermore, this property implies that a family of solutions which is locally uniformly bounded from below is compact.

In the next result we consider boundary value problems with L^1 boundary data.

THEOREM 1.2: *Suppose that g satisfies (1.4) and the Keller–Osserman condition.*

(i) *Assume that* Ω *is a smooth bounded domain,* $f \in (L^1_{loc})_+(\Omega)$ *and* $h \in L^1(\partial\Omega)$. *Then there exists a* **minimal supersolution** $\underline{u}_h \in L^1_{loc}(\Omega)$ *of the boundary value problem*

$$-\Delta u + g(u) = f \quad \text{in } \Omega, \quad u = h \quad \text{on } \partial\Omega. \tag{1.8}$$

The function $\underline{u}_h$ *satisfies (1.3) and, if* $f \in L^1(\Omega;\rho)$, *it is the unique solution of (1.8).*

(ii) *Assume that* Ω *is a bounded domain satisfying the classical Wiener condition,* $f \in (L^1_{loc})_+(\Omega)$ *and* $h \in C(\partial\Omega)$. *Then there exists a* **minimal supersolution** $\underline{u}_h \in L^1_{loc}(\Omega)$ *of (1.8). The function* $\underline{u}_h$ *satisfies (1.3) and, if* $f \in L^\infty(\Omega)$, *it is the unique solution of (1.8).*

For the definition of a supersolution of the boundary value problem (1.8) when f is only locally integrable, see Section 3. The definition of a sub/super solution of equation (1.3) is standard:

A function $u \in L^1_{loc}(\Omega)$ is a subsolution (resp. supersolution) of equation (1.3), with $f \in L^1_{loc}(\Omega)$, if $g(u) \in L^1_{loc}(\Omega)$ and

$$-\Delta u + g(u) - f \leq 0 \quad (\text{resp. } \geq 0) \quad \text{in } \Omega$$

in the distribution sense.

Remark: If u is a supersolution of equation (1.3), there exists a positive Radon measure μ in Ω such that

$$-\Delta u + g(u) - f = \mu, \quad \text{in } \Omega.$$

Therefore, (1.5) holds with f replaced by $f + \mu$:

$$\int_{\Omega'} (-u\Delta\phi + (g(u) - f)\phi)dx = \int_{\Omega'} \phi d\mu - \int_{\partial\Omega'} u\partial\phi/\partial\mathbf{n}'dS.$$

The following result concerns the existence of large solutions.

THEOREM 1.3: *Let* Ω *be a domain in* $\mathbb{R}^N$ *with nonempty, compact boundary. Assume that* g *satisfies (1.4) and the Keller–Osserman condition and that (1.1) possesses a subsolution* V *in* Ω. *Put*

$$\mathcal{U}_V(\Omega) := \{h \in L^1_{loc}(\Omega) : h \geq V \text{ a.e.}\}.$$

Under these assumptions:

(i) *For every* $f \in (L^1_{loc})_+(\Omega)$, *(1.3) possesses a minimal solution* V_f *in* $\mathcal{U}_V(\Omega)$. V_f *increases as* f *increases.*

(ii) *Assume, in addition, that Ω satisfies the (classical) Wiener criterion. Then, for every $f \in (L^1_{loc})_+(\Omega)$, (1.3) possesses a large solution. Moreover, there exists a* **minimal large solution** *of (1.3) in $\mathcal{U}_V(\Omega)$.*

(iii) *If Ω is bounded and satisfies the (classical) Wiener criterion, then (1.3) possesses a* **minimal large solution**.

Remark: (a) Part (i) implies that if (1.1) possesses a large solution, then (1.3) possesses a large solution for every $f \in (L^1_{loc})_+(\Omega)$. In [10] it was shown that, if g satisfies (1.4) and the so-called *weak singularity condition*, then (1.1) possesses a large solution in any domain Ω such that $\partial\Omega = \partial\bar{\Omega}^c$. The weak singularity condition is satisfied, for example, in the following cases:

(1) If $g(u) = |u|^{q-1}u$ and $1 < q < N/(N-2)$ for $N \geq 3$.

(2) If $0 < g(u) < ce^{au}$, $a > 0$, for $N = 2$.

(b) Labutin [6] studied power nonlinearities, $g(u) = |u|^{q-1}u$, $q > 1$, and showed that a necessary and sufficient condition for the existence of large solutions of (1.1) is that Ω satisfy a Wiener-type condition in which the classical capacity $C_{1,2}$ is replaced by the capacity $C_{2,q'}$. Labutin's condition is less restrictive than the classical Wiener condition; however, the latter applies to every nonlinearity satisfying the conditions of Theorem 1.3. It is interesting to know if the classical Wiener condition is necessary for the existence of large solutions under these general conditions. More precisely we ask:

Open problem 1: Let Ω be a bounded domain which does not satisfy the (classical) Wiener criterion at some point $P \in \partial\Omega$. Does there exist a function g satisfying (1.4) and the Keller–Osserman condition such that the maximal solution of (1.1) is not a large solution?

In continuation, we consider the question of uniqueness of large solutions, for nonlinearities g as in Theorem 1.3. In order to deal with this question in unbounded domains, we have to restrict ourselves to solutions which are essentially bounded below by a subsolution of (1.1).

THEOREM 1.4: *Let Ω be a domain in $\mathbb{R}^N$ with non-empty, compact boundary. Assume that g is convex and satisfies (1.4) and the Keller–Osserman condition.*

(i) *Let V be a subsolution of (1.1). If (1.1) possesses a unique large solution in $\mathcal{U}_V(\Omega)$, then, for every $f \in (L^1_{loc})_+(\Omega)$, (1.3) possesses a unique large solution in $\mathcal{U}_V(\Omega)$.*

(ii) *Let Ω be a bounded domain. If (1.1) possesses a unique large solution, then, for every $f \in (L^1_{loc})_+(\Omega)$, (1.3) possesses a unique large solution.*

(iii) *Suppose that Ω is a bounded domain such that $\partial\Omega$ is a locally continuous graph. Then (1.1) possesses at most one large solution.*
If, in addition, Ω satisfies the classical Wiener condition, then (1.3) possesses a unique large solution for every $f \in (L^1_{loc})_+(\Omega)$.

Remark: Equation (1.3) may possess one or more large solutions even if (1.1) does not possess such a solution. Furthermore, if Ω is unbounded, (1.3) may possess several large solutions even if (1.1) possesses a unique large solution. In fact, assertion (ii) implies that if (1.1) possesses a unique large solution W, then (1.3) possesses a unique large solution bounded below by W. However, if Ω is unbounded, (1.3) may possess additional large solutions which are not bounded below by W.

Finally, we present two results involving solutions in the whole space.

THEOREM 1.5: *Let $\Omega = \mathbb{R}^N$. Assume that g satisfies (1.4) and the Keller–Osserman condition and that (1.1) possesses a subsolution V. Then:*
(i) *For every $f \in (L^1_{loc})_+(\mathbb{R}^N)$, (1.3) possesses a solution u in $\mathcal{U}_V(\mathbb{R}^N)$.*
(ii) *Assume, in addition, that g is convex. If (1.1) possesses a* **unique solution** *in $\mathcal{U}_V(\mathbb{R}^N)$, then, for every $f \in (L^1_{loc})_+(\Omega)$, (1.3) possesses a* **unique solution** *in $\mathcal{U}_V(\mathbb{R}^N)$.*

For the statement of the next theorem we need the following notation. If g is a function defined on $\mathbb{R}$ such that $g(0) = 0$, we denote by $\tilde{g}$ the function given by $\tilde{g}(t) = -g(-t)$ for every real t.

THEOREM 1.6: *Assume $\Omega = \mathbb{R}^N$. Suppose that g and $\tilde{g}$ satisfy (1.4) and the Keller–Osserman condition. Then:*
(i) *For every $f \in L^1_{loc}(\mathbb{R}^N)$, (1.3) possesses a solution.*
(ii) *Assume, in addition, that g is convex in $(0,\infty)$ and $g(0) = 0$. Then, for every $f \in (L^1_{loc})_+(\mathbb{R}^N)$, (1.3) possesses a unique positive solution.*

Remark: It can be shown that if, in addition to the assumptions of part (ii), g satisfies the condition

$$g(a+b) \le c(g(a)+g(b)) \quad \forall a,b \in (0,\infty) \tag{1.9}$$

for some constant $c > 0$, then (1.3) possesses a unique solution in $\mathbb{R}^N$, for every $f \in L^1_{loc}(\Omega)$. (This condition implies that g is dominated by a power.) In the case that g is a power, this result was obtained by Brezis [3].

Open problem 2: Suppose that g is a function of exponential growth at $+\infty$, e.g.,

$$g(t) = (e^{(t^\alpha)} - 1)\,\mathrm{sign}\, t \quad \forall t \in \mathbb{R}^N$$

for some $\alpha > 0$. Does (1.3) possess a *unique* solution in $\mathbb{R}^N$, for every $f \in L^1_{loc}(\Omega)$?

2. Existence of a maximal solution

Proof of Theorem 1.1: Let $\{\Omega_n\}$ be a sequence of bounded subsets of Ω with smooth boundary such that

$$\Omega_n \uparrow \Omega, \quad \bar{\Omega}_n \subset \Omega_{n+1}. \tag{2.1}$$

For every $n \in \mathbb{N}$ and $m, k > 0$, denote by $u_{n,m,k}$ the classical solution of

$$-\Delta u + g(u) = f_k := \min(f, k) \quad \text{in } \Omega_n, \quad u = m \quad \text{on } \partial\Omega_n. \tag{2.2}$$

Further, denote by $v_{n,m}$ and $w_{n,k}$ the solutions of

$$-\Delta v + g(v) = 0 \quad \text{in } \Omega_n, \quad v = m \quad \text{on } \partial\Omega_n, \tag{2.3}$$

and

$$-\Delta w = f_k \quad \text{in } \Omega_n, \quad w = 0 \quad \text{on } \partial\Omega_n, \tag{2.4}$$

respectively. Then $u_{n,m,k} - v_{n,m} \geq 0$ and hence

$$-\Delta(u_{n,m,k} - v_{n,m}) = f_k - g(u_{n,m,k}) + g(v_{n,m}) \leq f_k.$$

Since $u_{n,m,k} - v_{n,m}$ vanishes on $\partial\Omega_n$, it follows that

$$u_{n,m,k} - v_{n,m} \leq w_{n,k} \quad \forall m \in \mathbb{N}. \tag{2.5}$$

Both $m \mapsto v_{n,m}$ and $m \mapsto u_{n,m,k}$ are increasing and $v_{n,m} \leq u_{n,m,k}$. Since g satisfies the Keller–Osserman condition, $\lim_{m\to\infty} v_{n,m} = v_n$ is the minimal large solution of (1.1) in Ω_n. Therefore, by (2.5),

$$v_n \leq u_{n,k} = \lim_{m\to\infty} u_{n,m,k} \leq v_n + w_{n,k}. \tag{2.6}$$

Since $w_{n,k}$ is bounded and v_n is locally bounded, it follows that $u_{n,k}$ is locally bounded in Ω_n. Thus $u_{n,k}$ is a large solution of (2.2), for every $k > 0$. Both $k \mapsto u_{n,k}$ and $k \mapsto w_{n,k}$ are increasing. Hence, letting $k \to \infty$, we obtain

$$v_n \leq u_n = \lim_{k\to\infty} u_{n,k} \leq v_n + w_n, \tag{2.7}$$

where w_n is the solution of

$$-\Delta w = f \quad \text{in } \Omega_n, \quad w = 0 \quad \text{on } \partial\Omega_n. \tag{2.8}$$

For every $\zeta \in C_c^2(\Omega_n)$,

$$\int_{\Omega_n} (-u_{n,k}\Delta\zeta + g(u_{n,k})\zeta)dx = \int_{\Omega_n} f_k\zeta dx.$$

Since $g(u_{n,k}) \uparrow g(u_n)$, $f \in L^1(\Omega_n)$ and, by (2.7) and (2.8), $u_n \in L^1_{loc}(\Omega_n)$, it follows that

$$\int_{\Omega_n} (-u_n\Delta\zeta + g(u_n)\zeta)dx = \int_{\Omega_n} f\zeta dx,$$

for every $\zeta \in C_c^2(\Omega_n)$, $\zeta \geq 0$. In addition, $u_n \geq v_n$ and consequently the negative part of u_n is bounded. Therefore, if $\Omega_n^+ = \Omega_n \cap \{u_n \geq 0\}$, we obtain

$$0 \leq \int_{\Omega_n^+} g(u_n)\zeta dx < \infty,$$

for every ζ as above. This implies that $g(u_n) \in L^1_{loc}(\Omega_n)$ and u_n is a large solution of (1.3) in Ω_n. Clearly, $\{u_n\}$ is monotone decreasing and $u_n \geq v_0$ in Ω_n for any subsolution v_0 of (1.1); by assumption, such a subsolution exists. Therefore, $u := \lim u_n$ is well defined and it is a solution of (1.3) in Ω. In fact, u is the maximal solution of (1.3) in Ω. Indeed, if U is a solution of (1.3) then, in view of (1.5), $U \leq u_n$ in Ω_n, so that $U \leq u$. ∎

3. Minimal supersolutions of boundary value problems

We start with the definition of a supersolution of (1.8) when f is only locally integrable.

Definition 3.1: Under the conditions of part (i) (resp. (ii)) of Theorem 1.2, a function $u \in L^1_{loc}(\Omega)$ is a supersolution of the *boundary value problem* (1.8) if it is a supersolution of (1.3) and, for every $f_0 \in L^1_+(\Omega)$ (resp. $f_0 \in L^\infty_+(\Omega)$) such that $f_0 \leq f$, u dominates the solution of the boundary value problem

$$-\Delta u + g(u) = f_0 \quad \text{in } \Omega, \quad u = h \quad \text{on } \partial\Omega.$$

Proof of Theorem 1.2: First we verify the following assertion:

If $u \in L^1_{loc}(\Omega)$ is a supersolution (in the sense of Definition 3.1) of the boundary value problems

$$-\Delta u + g(u) = f_k = \min(f,k) \quad \text{in } \Omega, \quad u = h \quad \text{on } \partial\Omega, \tag{3.1}$$

for every $k > 0$, then u is a supersolution of (1.8).

Under the assumptions of part (ii) the assertion is true by definition. Therefore, we assume the conditions of part (i). Let $\tilde{f} \in L^1_+(\Omega)$ be a function dominated by f and put $\tilde{f}_k := \min(\tilde{f}, k)$. If $\tilde{u}_k$ is the solution of (3.1) with f_k replaced by $\tilde{f}_k$, then $\tilde{u}_k \uparrow \tilde{u}$ where $\tilde{u}$ is the solution of

$$-\Delta\tilde{u} + g(\tilde{u}) = \tilde{f} \quad \text{in } \Omega, \quad \tilde{u} = h \quad \text{on } \partial\Omega.$$

By assumption, $\tilde{u}_k \leq u$, for every $k > 0$. Hence $\tilde{u} \leq u$ and the assertion is proved.

Denote by u_k the unique solution of (3.1). Since Ω is bounded, there exists a solution of (1.1). Therefore, by Theorem 1.1, there exists a maximal solution $\bar{u}_f$ of (1.3). Then $u_k \leq \bar{u}_f$ and $\{u_k\}$ is increasing. Consequently, $u = \lim u_k$ is a solution of (1.3) and, by the first part of the proof, it is a supersolution of (1.8). Obviously, it is the minimal supersolution of (1.8). ∎

4. Existence of a large solution

We recall that an open subset Ω of $\mathbb{R}^N$ satisfies the Wiener criterion if, for every $\sigma \in \partial\Omega$,

$$\int_0^1 C_{1,2}(B_s(\sigma) \cap \Omega^c)\frac{ds}{s^{N-1}} = \infty, \tag{4.1}$$

where $C_{1,2}$ stands for the classical (electrostatic) capacity. If Ω is a domain with compact, nonempty boundary and the Wiener criterion is fulfilled, then for any $\phi \in C(\partial\Omega)$ and $\psi \in L^\infty_{loc}(\overline{\Omega})$, a weak solution of

$$\begin{cases} -\Delta w = \psi & \text{in } \Omega \\ w = \phi & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

is continuous up to $\partial\Omega$.

Suppose that V is a subsolution of (1.1), i.e., V and $g(V)$ are in $L^1_{loc}(\Omega)$ and $-\Delta V + g(V)$ is a negative distribution. It follows that there exists a positive Radon measure μ such that

$$-\Delta V + g(V) = -\mu \quad \text{in } \Omega.$$

Consequently, $V \in W^{1,p}_{loc}(\Omega)$ for some $p > 1$ and, if Ω' is a smooth bounded domain such that $\bar{\Omega}' \subset \Omega$, then V possesses an L^1 trace on $\partial\Omega'$ and

$$\int_{\Omega'} (-V\Delta\phi + g(V)\phi)dx = -\int_{\Omega'} \phi d\mu - \int_{\partial\Omega'} V\partial\phi/\partial\mathbf{n}' dS, \tag{4.3}$$

for every $\phi \in C_0^2(\bar{\Omega}')$, where $\mathbf{n}'$ denotes the external unit normal on $\partial\Omega'$.

Proof of Theorem 1.3(i): Let V be a subsolution of (1.1) and let $\{\Omega_n\}$ be as in the proof of Theorem 1.1. Let $V_{f,n}$ be the (unique) solution of the problem

$$-\Delta u + g(u) = f \quad \text{in } \Omega_n, \quad u = V \quad \text{on } \partial\Omega_n. \tag{4.4}$$

Since V is a subsolution,

$$V_{f,n+1} \geq V_{f,n} \quad \text{in } \Omega_n. \tag{4.5}$$

By Theorem 1.1, there exists a maximal solution $\bar{u}_{f,n}$ (resp. $\bar{u}_f$) of (1.3) in Ω_n (resp. Ω). Clearly

$$\bar{u}_f|_{\Omega_n} \leq \bar{u}_{f,n+1}|_{\Omega_n} \leq \bar{u}_{f,n}. \tag{4.6}$$

Therefore, $\{\bar{u}_{f,n}\}$ converges and the limit U is a solution in Ω such that $U \geq \bar{u}_f$. As $\bar{u}_f$ is the maximal solution, it follows that $U = \bar{u}_f$; thus

$$\bar{u}_f = \lim_{n\to\infty} \bar{u}_{f,n}. \tag{4.7}$$

Since $V_{f,n} \leq \bar{u}_{f,n}$, (4.5) and (4.7) imply that the sequence $\{V_{f,n}\}$ converges to a solution V_f of (1.3). Clearly, V_f is the minimal solution in $\mathcal{U}_V$. By the maximum principle, $V_{f,n}$ increases with f. Therefore, V_f increases with f.

Proof of Theorem 1.3(ii): Let $\{\Omega_n\}$ be a sequence of domains contained in Ω satisfying (2.1) such that, for each $n \in \mathbb{N}$, $\Gamma_n = \partial\Omega_n$ is a smooth compact surface and if Ω is unbounded then Ω_n is also unbounded. In this last case, let $\{D_{n,j} : n, j \in \mathbb{N}\}$ be a family of smooth bounded domains such that

$$\bar{D}_{n,j} \subset D_{n+1,j+1}, \quad \partial D_{n,j} = \Gamma_n \cup \Gamma'_j, \quad \Gamma_n \cap \Gamma'_j = \emptyset,$$

where Γ_n and Γ'_j are smooth, compact surfaces and

$$\bigcup_{j\geq 1} D_{n,j} = \Omega_n.$$

Denote

$$\Omega'_j := \bigcup_{n\geq 1} D_{n,j}.$$

If Ω is bounded we put $D_{n,j} = \Omega_n$, $\Gamma'_j = \emptyset$ for every $j \in \mathbb{N}$ so that, in this case, $\Omega'_j = \Omega$.

Let V_0 be the minimal solution of (1.1) bounded below by V. Let $u^0_{m,n,j}$ be the solution of the problem

$$\begin{cases} -\Delta u + g(u) = 0 & \text{in } D_{n,j}, \\ u = \max(m, V_0) & \text{on } \Gamma_n, \\ u = V_0 & \text{on } \Gamma'_j. \end{cases} \tag{4.8}$$

By the maximum principle, $u^0_{m,n,j}$ increases with m and j and $u^0_{m,n,j} \geq V_0$. In addition, by the Keller–Osserman estimate, the set

$$\{u^0_{m,n,j} : m \geq 1, n > n_0, j > j_0\}$$

is bounded in D_{n_0,j_0}. Therefore, there exists a subsequence $\{n'\}$ such that the limit

$$z^0_{m,j} = \lim_{n'\to\infty} u^0_{m,n',j} \tag{4.9}$$

exists in Ω'_j, $z^0_{m,j}$ is a solution of (1.1) in this domain and

$$z^0_{m,j} \geq m \quad \text{on } \partial\Omega, \quad z^0_{m,j} = V_0 \quad \text{on } \Gamma'_j, \quad z^0_{m,j} \geq V_0 \quad \text{in } \Omega'_j. \tag{4.10}$$

In fact, if $w^0_{m,j}$ is the solution of the problem

$$\begin{cases} -\Delta w + g(w) = 0 & \text{in } \Omega'_j, \\ w = m & \text{on } \partial\Omega, \\ w = V_0 & \text{on } \Gamma'_j, \end{cases} \tag{4.11}$$

then $w^0_{m,j} \in C(\bar{\Omega}'_j)$. (Here, we use the fact that Ω satisfies the Wiener criterion.) In addition, for any $\delta > 0$, if n is sufficiently large, then Γ_n is contained in a δ-neighborhood of $\partial\Omega$. Therefore, $\sup w^0_{m,j}|_{\Gamma_n} \to m$ as $n \to \infty$ and $u^0_{m,n,j} \geq w^0_{m,j}$ for all sufficiently large n. Consequently, $z^0_{m,j} \geq w^0_{m,j}$. Further, if U is a large solution of (1.1) and $U \geq V_0$, then U dominates $u^0_{m,n,j}$ for all sufficiently large n. Hence $U \geq z^0_{m,j}$. Therefore,

$$\underline{u}^0_V := \lim_{j\to\infty} \lim_{m\to\infty} z^0_{m,j}$$

is the minimal large solution of (1.1) which dominates V_0 (and hence V). Consequently, if $\underline{u}^f_V$ denotes the minimal solution of (1.3) which dominates $\underline{u}^0_V$, then $\underline{u}^f_V$ is a large solution of (1.3) which dominates V. Further, if $U^f \in \mathcal{U}_V$ is a large solution of (1.3), then, for fixed $m, j \in \mathbb{N}$, $U^f \geq u^0_{m,n,j}$ for all sufficiently large n. Hence $U^f \geq z^0_{m,j}$, which in turn implies $U^f \geq \underline{u}^0_V$ and $U^f \geq \underline{u}^f_V$. Thus $\underline{u}^f_V$ is the minimal large solution of (1.3) in $\mathcal{U}_V$.

For later reference we observe that, for an appropriate choice of $\{D_{n,j}\}$,

$$\underline{u}_V^f = \lim_{j\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} u^f_{m,n,j}. \tag{4.12}$$

Of course, the family of domains $\{D_{n,j}\}$ can be chosen so that (4.12) holds for a given finite set of functions f.

Proof of Theorem 1.3(iii): Put $f_k := \min(f, k)$, $k \in \mathbb{N}$. Let $u_{k,m}$ be the (unique) solution of the problem,

$$\begin{aligned} -\Delta w + g(w) &= f_k \quad \text{in } \Omega, \\ w &= m \quad \text{on } \partial\Omega. \end{aligned} \tag{4.13}$$

Obviously, $u_{k,m} \le \bar{u}_f$ (=the maximal solution of (1.3)). Since $m \mapsto u_{k,m}$ is increasing, it follows that $u_k := \lim_{m\to\infty} u_{k,m} \le \bar{u}_f$ is a large solution of $-\Delta w + g(w) = f_k$ in Ω. Further, $k \mapsto u_k$ is also increasing. Thus $\underline{u}^f := \lim u_k$ is a large solution of (1.3). Every large solution U of (1.3) dominates $u_{k,m}$. Therefore, $\underline{u}^f$ is the minimal large solution. ∎

5. Uniqueness

Proof of Theorem 1.4(i): Let $\{D_{n,j}\}$ be as in the proof of Theorem 1.3, chosen so that (4.12) holds for both f and the zero function. In fact, we shall use all the notation introduced in this proof. Let $U^f_{m,n,j}$ be the solution of the problem

$$\begin{cases} -\Delta u + g(u) = f & \text{in } D_{n,j}, \\ w = \max(m, V_0) & \text{on } \partial D_{n,j}. \end{cases} \tag{5.1}$$

Then $U^f_{n,j} = \lim_{m\to\infty} U^f_{m,n,j}$ is a large solution of (1.3) in $D_{n,j}$ and

$$\bar{u}_f := \lim_{j\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} U^f_{m,n,j} \tag{5.2}$$

is the maximal solution of (1.3) in Ω. If (1.1) possesses a large solution, then, of course, $\bar{u}_f$ (resp. $\bar{u}_0$) is the maximal large solution of (1.3) (resp. (1.1)).

Put

$$Z^f = Z^f_{m,n,j} := U^f_{m,n,j} - u^f_{m,n,j} \le 0.$$

Then

$$\Delta(Z^f - Z^0) = g(U^f_{m,n,j}) - g(U^0_{m,n,j}) - g(u^f_{m,n,j}) + g(u^0_{m,n,j}),$$

in $D_{n,j}$. We rewrite the right-hand side in the form

$$\bar{d}_f(U^f_{m,n,j} - U^0_{m,n,j}) - \underline{d}_f(u^f_{m,n,j} - u^0_{m,n,j}),$$

where

$$\bar{d}_f = \frac{g(U^f_{m,n,j}) - g(U^0_{m,n,j})}{U^f_{m,n,j} - U^0_{m,n,j}}, \quad \underline{d}_f = \frac{g(u^f_{m,n,j}) - g(u^0_{m,n,j})}{u^f_{m,n,j} - u^0_{m,n,j}}.$$

Since g is convex and nondecreasing,

$$\bar{d}_f \geq \underline{d}_f \geq 0, \quad \Delta(Z^f - Z^0) \geq \underline{d}_f(Z^f - Z^0)$$

in $D_{n,j}$. As $Z^f - Z^0{=}0$ on $\partial D_{n,j}$, it follows that

$$Z^f - Z^0 \leq 0 \quad \text{in } D_{n,j}.$$

Thus

$$U^f_{m,n,j} - U^0_{m,n,j} \leq u^f_{m,n,j} - u^0_{m,n,j}$$

and, consequently,

$$\lim_{j\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} (U^f_{m,n,j} - U^0_{m,n,j}) \leq \lim_{j\to\infty} \lim_{m\to\infty} \lim_{n\to\infty} (u^f_{m,n,j} - u^0_{m,n,j}).$$

Hence, by (5.2) and (4.12),

$$\bar{u}_f - \bar{u}_0 \leq \underline{u}^f_V - \underline{u}^0_V.$$

Thus

$$0 \leq \bar{u}_f - \underline{u}^f_V \leq \bar{u}_0 - \underline{u}^0_V. \tag{5.3}$$

Assuming that (1.1) possesses a unique large solution dominating V, we find that $\bar{u}_0 = \underline{u}^0_V$ and hence $\bar{u}_f = \underline{u}^f_V$. Therefore, (1.3) possesses a unique large solution in the class of functions dominating V.

Proof of Theorem 1.4(ii): If (1.1) possesses a large solution U_0, then (1.3) possesses a large solution $U \geq U_0$. Since Ω is bounded, (1.3) possesses a minimal large solution $\underline{u}^f$ (by Theorem 1.3(iii)) and a maximal solution $\bar{u}_f$ (by Theorem 1.1). If U_0 is the unique large solution of (1.1), then, by the same argument as in part (i), $\underline{u}^f = \bar{u}_f$.

Proof of Theorem 1.4(iii): Since g is convex,

$$g(a+b) \geq g(a) + g(b) - g(0) \quad \forall a, b \geq 0.$$

Therefore, by [10, Theorem 0.3], (1.1) possesses at most one large solution. By Theorem 1.3(ii), (1.3) possesses a large solution for every $f \in (L^1_{loc})_+(\Omega)$. Therefore (iii) is a consequence of (ii). ∎

6. Solutions in the whole space

Proof of Theorem 1.5: (i) Let u^f_R be the maximal solution of (1.3) in $B_R = B_R(0)$; its existence is guaranteed by Theorem 1.1. If V is a subsolution of (1.1), $u^f_R \geq V$ and decreases with R. Hence $u^f = \lim_{R\to\infty} u^f_R$ is a solution of (1.3) in $\mathbb{R}^N$ and $u^f \geq V$.

(ii) Obviously, u^f is the maximal solution of (1.3) in $\mathbb{R}^N$. Next, we construct the minimal solution bounded below by V. For $R > 0$, let v^f_R be the solution of the problem

$$\begin{cases} -\Delta v + g(v) = f & \text{in } B_R, \\ v = V & \text{on } \partial B_R. \end{cases} \tag{6.1}$$

Then

$$V \leq v^f_R \leq u^f_R. \tag{6.2}$$

Since V is a subsolution, v^f_R increases with R. Therefore

$$V \leq v^f := \lim_{R\to\infty} v^f_R \leq u^f. \tag{6.3}$$

Clearly, v^f is the minimal solution of (1.3) bounded below by V.

As in the proof of Theorem 1.4 we obtain,

$$u^f - v^f \leq u^0 - v^0.$$

If (1.1) possesses a unique solution in $\mathbb{R}^N$, then $u^0 = v^0$ and consequently $u^f = v^f$. Thus (1.3) possesses a unique solution bounded below by V. ∎

Proof of Theorem 1.6:

A-priori Estimates: If u is a solution of (1.3) in $\mathbb{R}^N$, then $\tilde{u}(\cdot) = -u(-\cdot)$ satisfies

$$-\Delta\tilde{u} + \tilde{g}(\tilde{u}) = \tilde{f} \quad \text{in } \mathbb{R}^N, \tag{6.4}$$

where $\tilde{f}(x) = -f(-x)$.

For every $R > 0$, let U_R be the maximal solution of

$$-\Delta v + g(v) = |f| \quad \text{in } B_R. \tag{6.5}$$

By Theorem 1.3, U_R is a large solution. Clearly $R \mapsto U_R$ decreases as R increases. Therefore, $U = \lim_{R\to\infty} U_R$ is the maximal solution of

$$-\Delta v + g(v) = |f| \quad \text{in } \mathbb{R}^N.$$

Similarly, let W_R be the maximal solution of

$$-\Delta w + \tilde{g}(w) = |\tilde{f}| \quad \text{in } B_R, \tag{6.6}$$

so that $W = \lim_{R\to\infty} W_R$ is the maximal solution of

$$-\Delta v + \tilde{g}(v) = |\tilde{f}| \quad \text{in } \mathbb{R}^N.$$

If u is any solution of (1.3) in B_R, then $u \le U_R$ and $\tilde{u} \le W_R$ so that

$$\tilde{W}_R \le u \le U_R. \tag{6.7}$$

Existence: Let $k > 0$, put $f_k = \min(|f|, k)\operatorname{sign} f$, and denote by $W_{k,R}$ and $U_{k,R}$ the maximal solutions defined above, with f replaced by f_k. Then $W_{k,R}$ and $U_{k,R}$ are locally bounded and increase with k. Consequently, if $\{u_R^k : R > 0\}$ is a family of functions such that u_R^k is a solution of (1.3) in B_R, with f replaced by f_k, this family is locally uniformly bounded. This means that, for every compact set K, there exists $R_k(K) > 0$ such that $\{u_R^k : R > R_k(K)\}$ is uniformly bounded in K. Therefore, there exists a sequence $R_j \to \infty$ such that $\{u_{R_j}^k\}$ converges locally uniformly to a solution u^k of (1.3) in $\mathbb{R}^N$, with f replaced by f_k. By (6.7), the family of solutions $\{u^k : k > 0\}$ is dominated (in absolute value) by a function in $L^1_{loc}(\mathbb{R}^N)$ and it is nondecreasing. Consequently, $u = \lim u^k$ is a solution of (1.3) in $\mathbb{R}^N$.

Uniqueness: Under the assumptions of (ii), $u \equiv 0$ is a solution of (1.1) in $\mathbb{R}^N$ and it is easy to see that this is the only solution. Therefore, the uniqueness statement for (1.3) follows by the same argument as in the proof of Theorem 1.5. ∎

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